DOCUMENT RESUME

ED 294 745 SE 049 139

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TITLE NCTM Student Math Notes 1987.
INSTITUTION National Council of Teachers of Mathematics, Inc., Reston, Va.
PUB DATE 87
NOTE 21p.; For 1986 Math Notes, see ED 276 567.
PUB TYPE Collected Works - Serials (022) -- Guides - Classroom Use - Materials (For Learner) (051)
JOURNAL CIT NCTM Student Math Notes; Jan-Nov 1987

EDRS PRICE MF01/PC01 Plus Postage.
DESCRIPTORS *Enrichment Activities; *Geometric Concepts; Junior High School Students; *Mathematics Instruction; *Number Concepts; Probability; Secondary Education; *Secondary School Mathematics

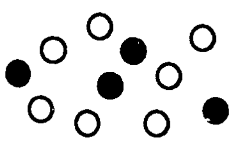
ABSTRACT

Five sets of activities for students are included in this document. Each is designed for use in junior high and secondary school mathematics instruction. The first is concerned with the use of models in mathematics. The second contains a variety of activities involving polygonal numbers. The third set of activities includes a variety of games for which the students are to decide if the game is fair (a game for which a player has a fifty-fifty chance of winning). The fourth set of activities is related to geometry with an emphasis on circles. The fifth set of activities involves punching paper to explore numerical patterns. (RH)

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
STUDENT math notes
JANUARY 1987

Models: The Mathematician's Laboratory Equipment

A real-world happening or an abstract idea can be represented by diagrams, sentences, numerals, equations, and graphs. These are called *models*. Can you match the statement on the right with the appropriate model on the left?

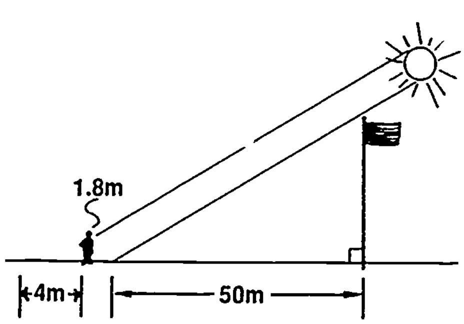
a. 

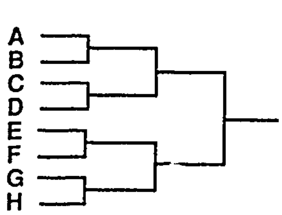
b. 

c. V, 5, 卌

d. 

e. $2n - n + 10 - n = ?$

f. 

g. 

____ 1. Water temperature when the hot-water faucet is turned on and left on

____ 2. I am thinking of a number. Twice the number minus the number plus 10 minus the number is what?

____ 3. A method of determining the games and winning team in a single-elimination tournament

____ 4. Fiveness

____ 5. Model of the ratio 4 to 7

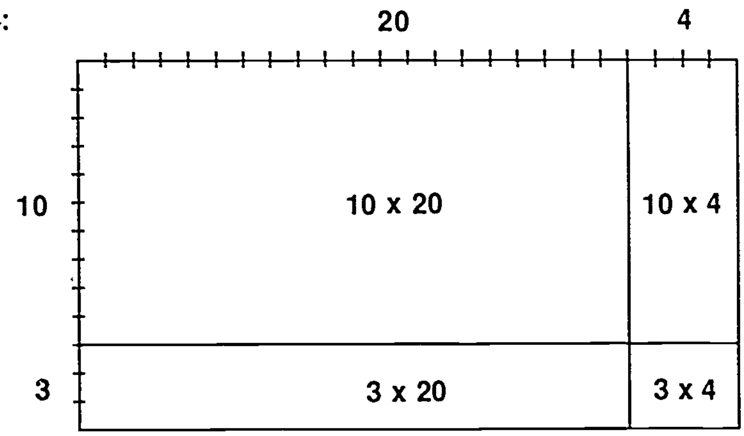
____ 6. An indirect method of measuring height

____ 7. 40 percent

The editors wish to thank Glenn Allinger and Lyle Andersen, Montana State University, Bozeman, MT 59717, for writing this issue of the *NCTM Student Math Notes.*

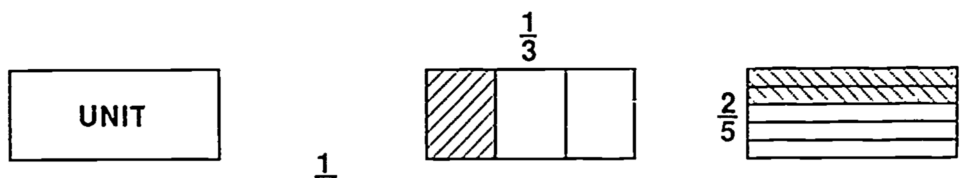
You use models to—

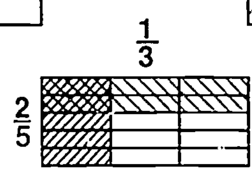
- find 13×24:



$$\begin{array}{r} 13 \\ \underline{\times\ 24} \\ 12 \\ 40 \\ 60 \\ \underline{200} \\ 312 \end{array}$$

- calculate the product of 2/5 and 1/3:

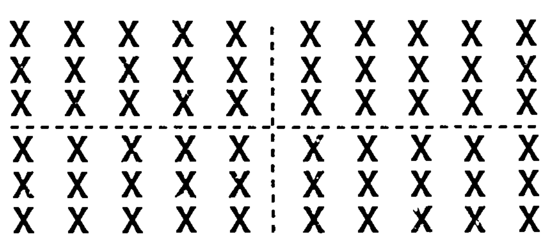




15 Total regions and 2 are shaded (▩), so $\frac{2}{5}$ of $\frac{1}{3}$ is $\frac{2}{15}$.

- determine 25 percent of 60:

25 percent of 60 is $\frac{1}{4}$ of 60.



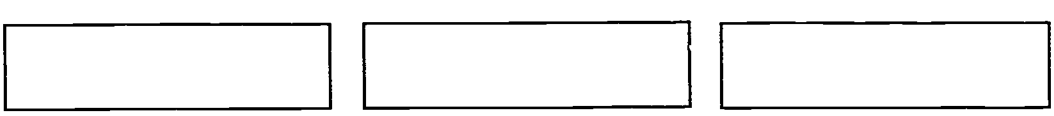
So $\frac{1}{4}$ of 60 is 15.

Related problems

Sketch a model and use it to calculate the following products:

8. Total number of people in 34 rows of 29 people each

9. 3/4 of 5/7



Show $\frac{3}{4}$ Show $\frac{5}{7}$ Show $\frac{3}{4}$ of $\frac{5}{7}$

10. A ski shop advertises 10 percent off on a pair of thermal gloves originally priced at $40. What is the dollar discount?

11. $(x + 3)(x + 4)$

3

Many Problems Have the Same Model

Sometimes one problem seems very different from another problem, yet the models used to help determine the solution are basically the same.

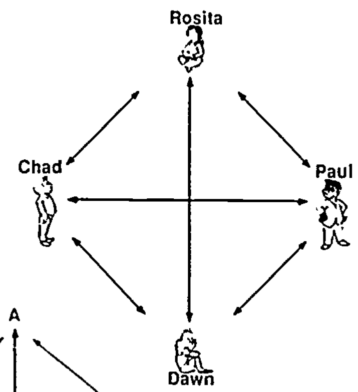
Example 1: Rosita enters a room in which Chad, Dawn, and Paul are already present. Everyone in the room shakes hands with everyone else. How many total handshakes will take place?

Rosita, Paul
Rosita, Dawn
Rosita, Chad

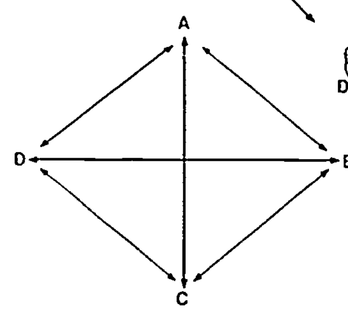
Paul, Dawn
Paul, Chad

Dawn, Chad

or



Example 2: Four sixth graders decide to help clean up the school grounds. They form subcommittees of two people each, with each subcommittee responsible for recycling a certain material, for example, metal, paper, glass, and chemicals. How many different subcommittees are possible?



or

Subcommittees		
A, B	B, C	C, D
A, C	B, D	
A, D		

Related problems

12. Five Girl Scouts from different troops meet for a leadership conference. In introducing themselves to each other, every girl shakes the hand of each of the other scouts. How many total handshakes are made?

13. From a group of five aerobic exercises, you can choose to do any three during the next free exercise period. The exercises require the stretching of muscles in the ankle, thigh, calf, shoulders, or neck. How many different groups of three exercises are possible?

Did You Know That...

- in the time of Columbus, many people thought that a flat plane was a good model for representing the earth?
- every time you solve a mathematical problem with pencil and paper you are building a model?
- the game of chess is a model of ancient war games?
- physicists think that the geometric model best suited to represent the universe is not Euclidean but elliptic, in which there are no parallels through a point outside a line?
- computers can produce photographic models using digits? The following digits represent a portion of a weather-satellite photograph transmitted by short-wave radio: 18462525747838383 . . .
- Japanese fishing crews use computer simulations to determine where the best fishing will be?
- meteorologists use the equation $D^3 = 216T^2$ as a model to describe the size and intensity of four types of violent storms: tornadoes, thunderstorms, hurricanes, and cyclones? D is the diameter of the storm in miles and T is the number of hours the storm travels before dissipating.

Can You...

14. use the equation developed by meteorologists and your calculator to answer the following?

a. The world's worst recorded monsoon took place on 13–14 November 1970 in the Ganges delta islands in Bangladesh. More than 1 000 000 people died. If this storm lasted 24 hours, what was its diameter?

b. In the U.S., about 150 tornadoes occur each year, mostly in the central plains states, with the highest frequency in Iowa, Kansas, Arkansas, Oklahoma, and Mississippi. If a tornado's diameter is 2 miles, how long would it be expected to last?

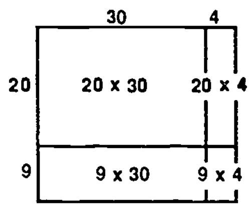
15. make a model to represent the product $(2x + 1)(3x + 2)$? (*Hint:* $x \square x$ represents x^2.)

16. Create a model to predict the minimum number of moves necessary for *K* boys sitting in the front half of a canoe to exchange seats with *K* girls sitting in the back half of a canoe with one empty seat between them given the following conditions: (*a*) *K* is an even natural number. (*b*) Each seat can only hold one person. (*c*) The only legal moves are to an adjacent empty seat or to step over one person to an empty seat. (*d*) The canoe will always have one more seat than people. When the moves are finished, the boys will have moved to the back half of the canoe and the girls to the front half. The seat in the middle will be empty.

Solutions

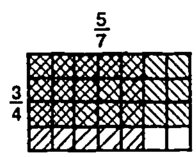
1. *d*; 2. *e*; 3. *g*; 4. *c*; 5. *a*; 6. *f*; 7. *b*

8.



$$\begin{array}{r} 29 \\ \times 34 \\ \hline 36 \\ 80 \\ 270 \\ 600 \\ \hline 986 \end{array}$$

9.



$$\frac{3}{4} \times \frac{5}{7} = \frac{15}{28}$$

10. \$:\$:\$:\$:\$:\$:\$:\$:\$
\$:\$:\$:\$:\$:\$:\$:\$:\$
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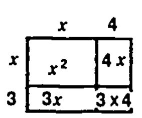
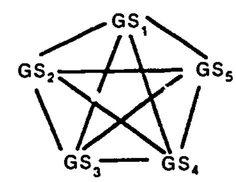
11.

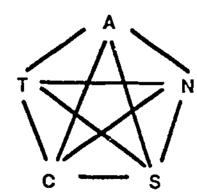


$$x^2 + 3x + 4x + 12$$
$$= x^2 + 7x + 12$$

12. 10 handshakes:

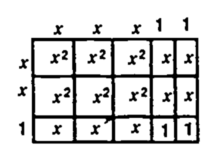


13. 10 different groups:



14. *a*. Approximately 50 miles; *b*. about 0.19 hours, or 11.5 minutes

15.



$6x^2 + 7x + 2$

16. When $K = 2$, the solution will have three moves (G stands for girls, B for boys): G __ B, GB __ , __ BG, B __ G. When $K = 4$, the solution will have eight moves. GG __ BB, GGB __ B, G __ BGB, __ GBGB, BG __ GB, BGBG __ , BGB __ G, B __ BGG, BB __ GG. When $K = 6$, the solution will have fifteen moves. When $K = 8$, the solution will have twenty-four moves. In general, for any *K*, the minimum number of moves will be $(1/4)K^2 + K$.

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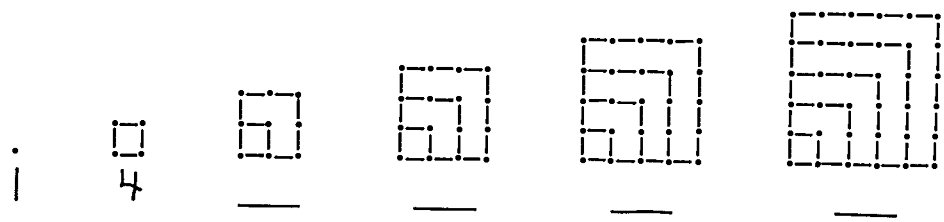
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
STUDENT math notes
MARCH 1987

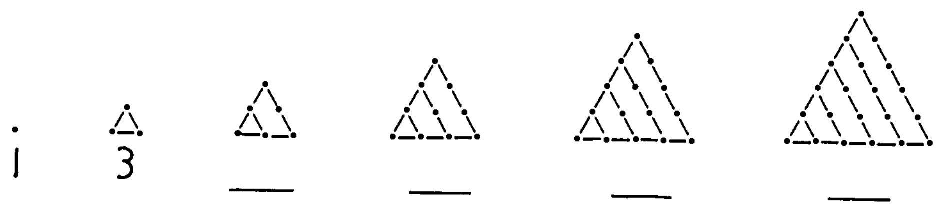
Polygonal Numbers

Mathematicians have given names to sets of numbers. You may know some of them: whole numbers, integers, natural numbers, real numbers, irrational numbers, and so on. Some numbers are associated with polygons and have geometric names.

The dots in the following figures represent *geometric numbers*. Fill in the blanks and determine each number.



What is the geometric shape of each of these figures? ____________



What is the geometric shape of each of .hese figures? ____________

The numbers determined by the shapes in the first group are called *square numbers*. The numbers determined by the shapes in the second group are called *triangular numbers*.

Mathematicians use abbreviations or symbols to represent geometric numbers. For example, we write S_4 to represent the fourth square number, a square with four dots on each side. Similarly we can write T_3 to represent the third triangular number, a triangle with three dots on each side. Using this notation, we can write $S_5 = 25$ and $T_3 = 6$.

Several interesting patterns can be explored involving triangular and square numbers, so you will need to build a table to show what some of these numbers are.

n	1	2	3	4	5	6	7	8	9	10	11	12
S_n	1	4	9	16	25	36						
T_n	1	3	6	10	15	21						

n	13	14	15	16	17	18	19	20	21	22	23	24
S_n												576
T_n												300

The editors wish to thank Boyd Henry, College of Idaho, Caldwell, ID 83605, for writing this issue of the *NCTM Student Math Notes*.

Explorations with Triangular Numbers

- Find the sum of any two consecutive triangular numbers. For example, suppose we pick T_6 and T_7; 21 + 28 = 49. Their sum is 49, the seventh square number. Add T_8 and T_9. Is the sum of these two consecutive triangular numbers a square number? What do you think you can say about the sum of *any* two consecutive triangular numbers?
- Draw an array to represent the seventh square number, S_7.
  Can you separate it into two consecutive triangular numbers?
- Complete the table below.

	n	1	2	3	4	5	6	7	8	9	10
A	T_n	1	3	6	10						
B	$8 \times T_n$	8	24	48							
C	$[8 \times T_n] + 1$	9	25	49							

What kind of numbers are in row C? In general, if we select any triangular number, multiply it by 8, and add 1, what is true about the result? ______

Further Explorations

- Let's investigate the nature of fractions that have numerators and denominators consisting of consecutive square numbers and consecutive triangular numbers. But first, complete the following table.

n	1	2	3	4	5	6	7	8	9
S_n/S_{n+1}	1/4	4/9	9/16						
T_n/T_{n+1}	1/3	3/6	6/10						

Look carefully at the fractions in the first row. All the numerators and denominators are square numbers. Can any of these fractions be reduced to lower terms?

Now examine the fractions in the second row, in which the numerators and denominators are all triangular numbers. The first fraction, 1/3, cannot be reduced. But what about the other fractions? Can all of them be reduced to lower terms?

Continuing Our Explorations

- As you complete the following table, consider these questions:

  What happens when a number is added to its square and the sum is divided by 2?
  (This is simply the average of any number and its square.)

  What happens when a number is subtracted from its square and the difference is divided by 2?

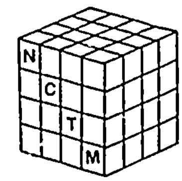
	n	2	3	4	5	6	7	8	9	10	11	12
	S_n	4	9	16								
	$S_n + n$	6	12	20								
A	$(S_n + n)/2$	3	6	10								
	$S_n - n$	2	6	12								
B	$(S_n - n)/2$	1	3	6								

- What do you notice about the numbers in rows A and B? ______

Explorations with Cubical Arrays

- We have already observed that a square number can be represented with a square array. In a similar manner, we can represent numbers called *cubes* with a cubical array. To build a cubical array measuring four units on each side, we need sixty-four units, since each of the four layers of the cube is a square array measuring 4×4. If we use the notation C_n to mean the cube of n, then $C_n = n^3$. Completing this table will help you recognize numbers that are cubes.

$C_4 = 4 \times 4 \times 4 = 4^3$, or 64



n	1	2	3	4	5	6	7	8	9	10	11	12
C_n	1	8	27	64								

- Now we will explore both the sum and difference of the squares of two consecutive triangular numbers. But first you must complete the following table.

	n	1	2	3	4	5	6	7	8	9	10
	T_n	1	3	6							
	T_{n+1}	3	6	10							
A	$[T_n]^2$	1	9	36							
B	$[T_{n+1}]^2$	9	36	100							
C	Row B + Row A	10	45	136							
D	Row B − Row A	8	27	64							
E	S_{n+1}	4	9	16							
F	$T_{S_{n+1}}$	10	45	136							

- In row C we have the sums of the squares of consecutive triangular numbers. Is each entry in row C a triangular number? __________ We know that the entries in row F are triangular numbers; in fact the entries in row F are triangular numbers associated with square numbers. Compare row C with row F. What do you conclude about the sum of the squares of two consecutive triangular numbers? __________

- In row D we have the difference between the squares of two consecutive triangular numbers. What kind of numbers do we have in row D? __________

- Let's explore some other relationships by completing this table.

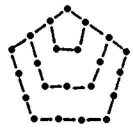
	n	1	2	3	4	5	6	7	8	9	10
A	C_{S_n}	1	64	729							
B	S_{C_n}	1	64	729							
C	T_{S_n}	1	10	45							
D	S_{T_n}	1	9	36							
E	Row A − Row B	0	0	0							
F	Row C − Row D	0	1	9							

- What conclusions can you draw from row E? __________

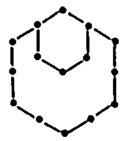
- What conclusions can you draw from row F? __________

Did You Know That...

- There are pentagonal and hexagonal numbers as well as triangular and square numbers? In fact numbers exist for each polygon. They are called *polygonal* or *figurate* numbers. Many interesting patterns can be found among these numbers.



The pentagon of 4



The hexagon of 3

- This BASIC program will compute any polygonal or figurate number you desire.

```
10  PRINT "THIS PROGRAM LISTS THE N-GON OF EACH NUMBER FROM 1 TO K"
20  INPUT " SELECT N ( ANY WHOLE NUMBER 3 OR GREATER).";N
30  INPUT "HOW FAR DO YOU WANT THE LIST TO GO";K
40  FOR A = 1 TO K
50  PRINT A;TAB(15);((N-2)*A*A-(N-4)*A)/2
60  NEXT A
70  END
```

Can You . . .

- Find patterns among the polygonal numbers by completing the following table?

n	1	2	3	4	5	6	7	8	9	10
Triangle: $T_n = n(n + 1)/2$	1	3	6							
Square: $S_n = n^2$	1	4	9							
Pentagon: $P_n = n(3n - 1)/2$	1	5	12							
Hexagon: $H_n = n(4n - 2)/2$	1	6	15							
Heptagon: $HP_n = n(5n - 3)/2$	1	7	18							
Octagon: $O_n = n(6n - 4)/2$	1	8	21							

- Show that . . .

S_n/S_{n+1} for $n \geq 1$ cannot be reduced to lower terms?

T_n/T_{n+1} for $n > 1$ can always be reduced to lower terms?

- Prove by mathematical induction that . . .

$T_n + T_{n+1} = S_{n+1}$ for $n \geq 1$?

$(S_n + n)/2 = T_{n+1}$ for $n \geq 2$?

$[T_n]^2 + [T_{n+1}]^2 = T_{S_{n+1}}$ for $n \geq 1$?

$[8T_n + 1] = (2n + 1)^2$ for $n \geq 1$?

$(S_n - n)/2 = T_n$ for $n \geq 2$?

$[T_{n+1}]^2 - [T_n]^2 = C_{n+1}$ for $n \geq 1$?

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NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
STUDENT math notes
MAY 1987

Fair Games

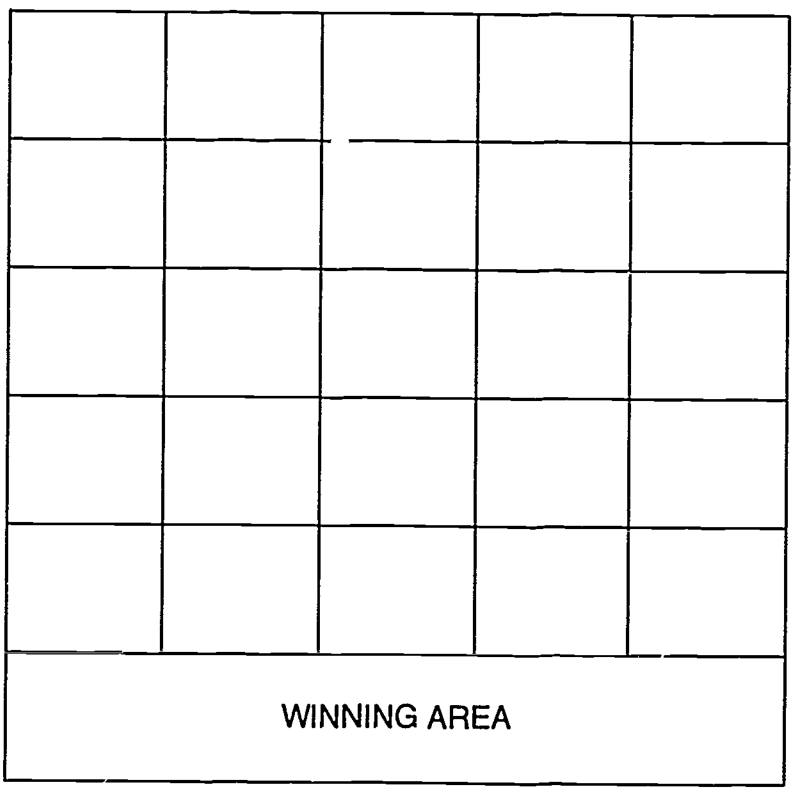
Playing games is a frequent source of entertainment. We enjoy the stimulation, the challenge to find winning strategies, and the competition. The following activities will allow you to explore three different games involving two players.

Game 1

Use a marker and the game board below. The rules for play are as follows:

1. Player A places the marker in any one of the empty cells in the top row.
2. Player B moves the same marker one cell to the right, one cell to the left, or one cell straight down. A player is *not* allowed to move up or diagonally.
3. Players alternate turns.
4. A player is *not* allowed to move to the previously occupied cell.
5. The first player who moves the marker into the winning area wins the game.

Play the game several times with another student.



Can you find a winning strategy? Does one appear to gain any advantage by being first or second? Analyze the game together with another student. Try to find a winning strategy. Find a way so that you can win every time.

Analyzing Fairness of Games

Definition: A game is fair if each player has a fifty-fifty chance of winning, or the probability of winning is one-half.

Game 2 (Version 1)

Here is a very basic but fun game for two people.

To play the game one person is called EVEN and the other is called ODD. Each person secretly writes down a number from *0 through 9* and covers the number. Next show the numbers. If the sum of the two numbers is even, the EVEN player wins. If the sum is odd, the ODD player wins. For example, if the two numbers written are 7 and 5, then the sum is 12, so the EVEN player wins.

		Total Wins
EVEN		
ODD		

Play the game several times, keeping track of wins by recording tally marks in the chart.

- Is this a fair game? ________ Argue your position.

Game 2 (Version 2)

A second version of the game is played *without 0*; thus a person can choose only numbers *1 through 9*. Play the game as described above.

		Total Wins
EVEN		
ODD		

- Is this a fair game? ________
- Who has the mathematical advantage, EVEN or ODD? ________

Definition: The probability of winning is the number of favorable outcomes divided by the total number of possible outcomes.

- Let's determine the probability that ODD will win. Fill in the addition table. Note that the usual order of headings 1, 2, 3, 4, 5, 6, 7, 8, 9 has been rearranged for convenience. From the table we have 81 possible outcomes. Counting gives 40 with an odd sum. Hence the probability that ODD will win is 40/81.
- Let's analyze the probability another way. ODD can win in two ways:

  Case 1: EVEN can choose an odd number and ODD can choose an even number. These outcomes are shown in the upper-right-hand corner of the table.

  Case 2: EVEN can choose an even number and ODD can choose an odd number. These outcomes are shown in the lower-left-hand corner of the table.

		Player ODD								
		Odd					Even			
	+	1	3	5	7	9	2	4	6	8
Player EVEN: Odd	1									
	3									
	5									
	7									
	9									
Player EVEN: Even	2									
	4									
	6									
	8									

ADDITION TABLE

Let's look further into case 1. The first five rows in the table identify those outcomes where EVEN chose an odd number. This total constitutes 5/9 of 81, or 45, outcomes. The last four columns identify those outcomes where ODD chose an even number. Hence, only 4/9 of EVEN's 45 outcomes satisfy case 1; that is, only 4/9 of 45, which is 20. Therefore, the probability that case 1 will occur is $4/9 \cdot 5/9$, or 20/81.

A parallel argument is used for case 2. The last four rows in the table identify those outcomes where EVEN chose an even number. This amount constitutes 4/9 of 81, or 36, outcomes. The first five columns identify those outcomes where ODD chose an odd number. Hence, only 5/9 of EVEN's 36 outcomes satisfy case 2; that is, 5/9 of 36, which is 20. Therefore, the probability that case 2 will occur is $5/9 \cdot 4/9$, or 20/81.

Since case 1 and case 2 are separate cases, the probability that ODD will win is $4/9 \cdot 5/9 + 5/9 \cdot 4/9 = 20/81 + 20/81 = 40/81$.

- What is the probability that EVEN will win? ________

Game 2 (Version 3)

This game can also be played with the same numbers (1–9); however, find the product of the two numbers chosen instead of the sum.

- Is this version a fair game? ________
- If you have a choice would you prefer to be EVEN or ODD? ________

Game 2 (Version 4)

By now you should realize that EVEN has a definite advantage in version 3. In fact, EVEN will always win by writing down any even number, since the product of two even numbers or of an even number and an odd number is always even.

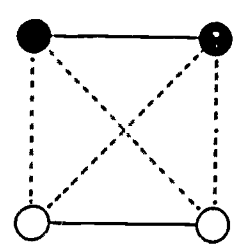
To prevent this eventuality, assume that a number from 1 through 9 is randomly assigned to you and your friend; that is, you no longer get to pick your number.

- What is the probability that the product is odd? ________
- What is the probability that the product is even? ________
- Is the game now "fair"? ________

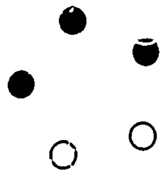
Game 3

Suppose that two white marbles and two black marbles are in a box. Without looking in the box, randomly choose two of the four marbles. If the two marbles are the same color, player A wins. If the two chosen marbles are each of a different color, player B wins. Play this game several times with another student.

- Is this game fair? ________ Why?



The model should convince you that it is not a fair game. It shows that six different ways exist to draw two of the four marbles. The solid lines show that only two ways exist to draw two marbles of the same color. The dotted lines show that four ways exist to draw two marbles of a different color. "Different" is favored to win by a two-to-one advantage over "same."



·· ' shows three black marbles and two white marbles. Draw in solid lines for . 'aw in dotted lines for "different."

..s the probability that "same" will win? ________

- V . is the probability that "different" will win? ________

- Many combinations of black and white marbles will produce a fair game. Can you find a combination to make it a fair game? ________
- Can you find other combinations that will make it a fair game? ________

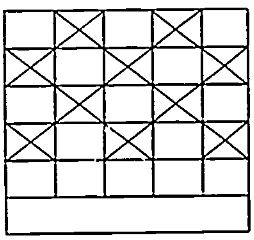
Can You...

- write a computer program that will simulate the various versions of game 2 to generate the empirical probabilities?
- write a computer program that will simulate game 3 for various combinations of black and white marbles?

Did You Know That...

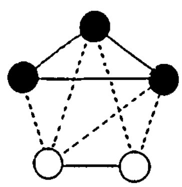
- games constitute an ideal model system in the development of artificial intelligence?
- two Frenchmen, Pierre de Fermat and Blaise Pascal, laid the foundations of probability theory through the analysis of various gambling games popular in France in the seventeenth century?
- archeologists uncovered, at prehistoric sites, large numbers of bones called *astragalia* that were apparently used as dice in ancient games?
- in England the game of American checkers is called *draughts*? Polish checkers is played on a ten-by-ten board with twenty pieces per side, rather than the usual twelve pieces. Turkish checkers is played with sixteen pieces per side.
- Johannes Gutenberg, inventor of the printing press, printed playing cards in 1440, the same year he printed his famous Gutenberg Bible?
- The game of chess was first played in Asia and was made popular in the United States by Benjamin Franklin?

Answers for bulleted questions: Game 1: You have to avoid the bottom row. By working backward from the bottom up you will also conclude that those cells marked with an X are safe. If you are first, place the marker on one of the two safe positions on the top row. As long as you avoid the last row of cells, you have no choice but to win, like it or not.



Game 2 (Version 2): Not fair, but very close. EVEN, but not by much. $P(\text{EVEN}) = 5/9 \cdot 5/9 + 4/9 \cdot 4/9 = 25/81 + 16/81 = 41/81$, or $1 - 40/81 \doteq 0.51$.

Game 2 (Version 4): $P(\text{ODD}) = 5/9 \cdot 5/9 = 25/81 \doteq 0.31$; $P(\text{EVEN}) = 4/9 \cdot 4/9 + 4/9 \cdot 5/9 + 5/9 \cdot 4/9 + 16/81 + 20/81 + 20/81 = 56/81 \doteq 0.69$. EVEN has the advantage.



Game 3: $P(\text{same}) = 4/10 = 0.4$; $P(\text{different}) = 1 - 4/10 = 6/10 = 0.6$. One black and three white is a fair game. Obviously three black and one white would also be a fair game. An infinite number of combinations will make it a fair game. For example, sixteen marbles of which six are black and ten are white would mathematically be a fair game.

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student math notes

Geometry in a Circle

Two friends, Tasha and Hank, were trying to determine where to put a point C on a major arc AB (larger than a semicircle) to form the largest possible angle ACB. Tasha picked C_1; Hank picked C_2.

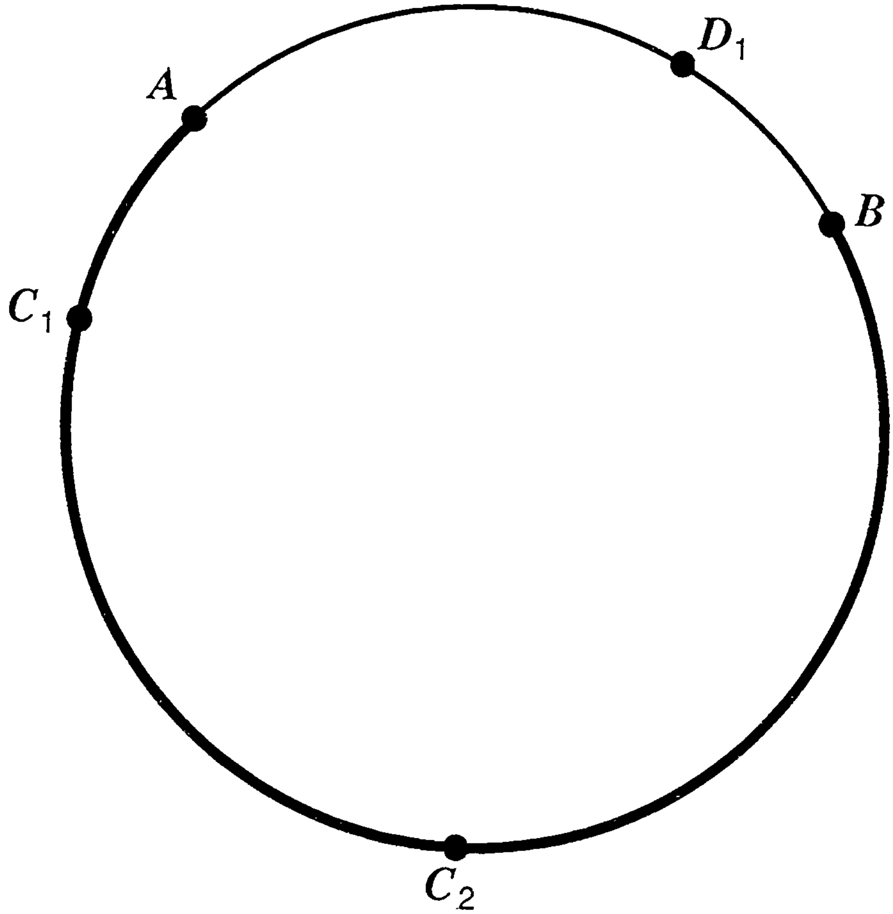
Who do you think made the better choice? ______________________

Why? ____

Construct and measure $\angle AC_1B$ and $\angle AC_2B$.

Pick two more points C_3 and C_4 on major arc AB.

Construct and measure $\angle AC_3B$ and $\angle AC_4B$.



1. $m\ \angle AC_1B =$ ______, $m\ \angle AC_2\mathrm{B} =$ ______, $m\ \angle AC_3B =$ ______, $m\ \angle AC_4B =$ ______
2. What appears to be true about the measures of these angles?

____

Sherrill thought she could make a larger angle by placing a point D_1 on the minor arc AB (less than a semicircle). Construct and measure $\angle AD_1B$. Pick another point on the minor arc AB and label it D_2. Construct and measure $\angle AD_2B$.

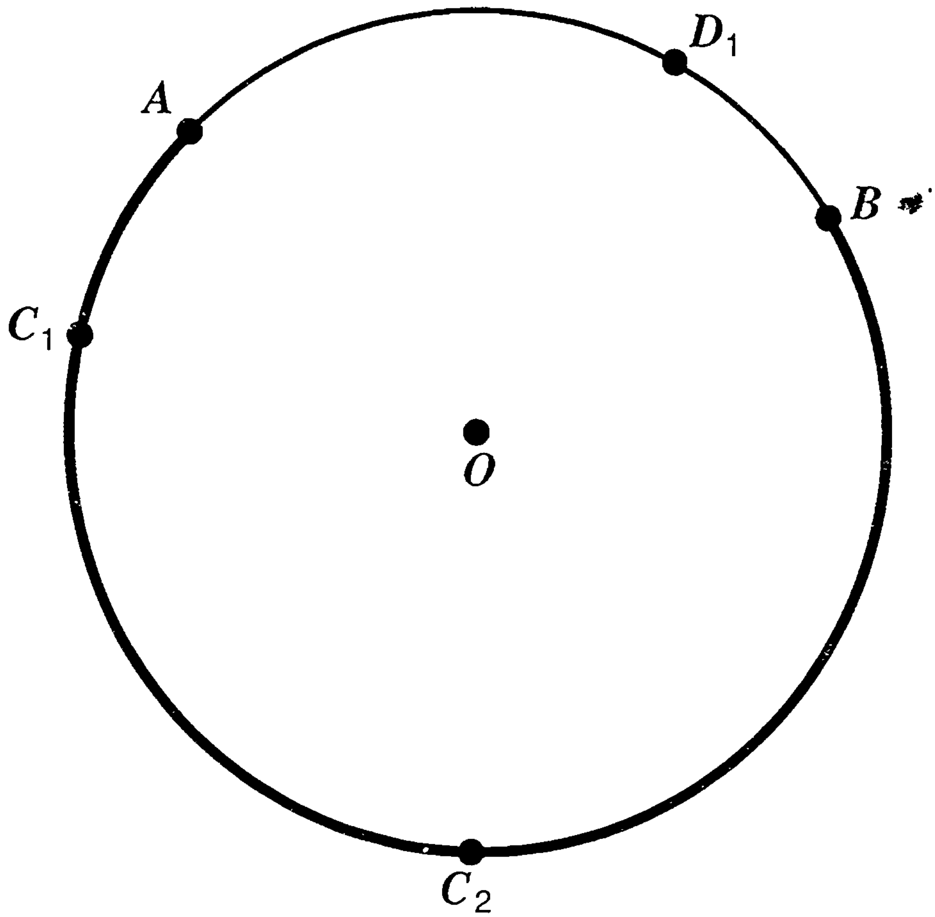
The editors wish to thank Melfried Olson and Judy Olson, Western Illinois University, Macomb, IL 61455, for writing this issue of *NCTM Student Math Notes*.

3. $m\ \angle AD_1B =$ ______, $m\ \angle AD_2B =$ ______

4. What appears to be true about these angles? __________________________________

5. What is the relationship between the measures of $\angle AC_1B$ and $\angle AD_1B$ or $\angle AC_2B$ and $\angle AD_2B$? ______

____



Pete disagreed with the others and thought it might be better to choose the center of the circle, O. Construct and measure $\angle AOB$.

6. $m\ \angle AOB =$ ______

7. How does $m\angle AC_1B$, $m\angle AC_2B$, $m\angle AC_3B$, or $m\angle AC_4B$ compare with $m\angle AOB$?

____

Draw a line segment from C_1 through O to the other side of the circle. Label the other endpoint F.

8. What is the segment C_1F called? ________________________________

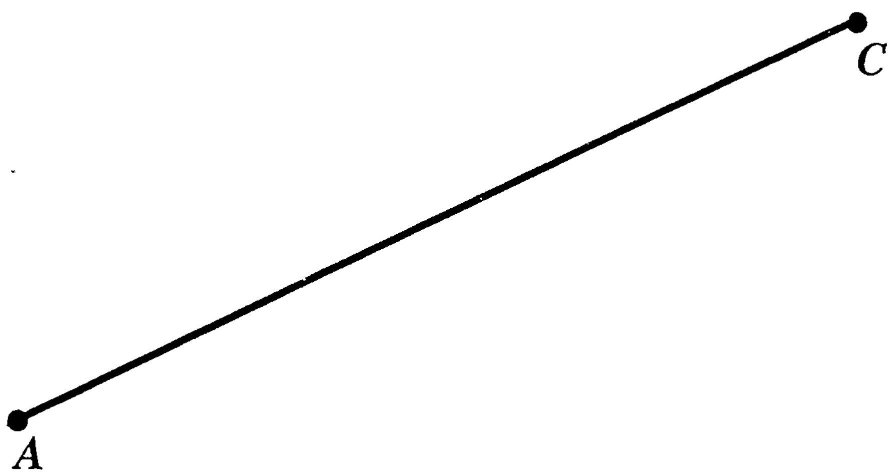
9. What is the arc C_1F called? ________________________________

10. What is the measure of the arc C_1F? ________________________________

Construct and measure $\angle C_1AF$, $\angle C_1D_1F$, and $\angle C_1BF$.

11. $m\angle C_1AF =$ ______, $m\angle C_1D_1F =$ ______, $m\angle C_1BF =$ ______

12. What appears to be true about any angle inscribed in a semicircle? ____________________



Use a ruler, protractor, or compass to complete the following construction:

- Draw a horizontal line ℓ through A.
- Locate a point P_1 on ℓ to form right triangle AP_1C.
- Draw another line ℓ_1 through A, below $\overline{AC}$, and locate P_2 on ℓ_1 to form right triangle AP_2C.
- Locate several other points P_i below $\overline{AC}$ such that $\triangle AP_iC$ is a right triangle.

13. What figure starts to appear if you look at A, C, and several points P_i? ______

- Locate point O, the midpoint of $\overline{AC}$.
- Draw segment P_1O and extend it to point Q_1 so that $Q_1O = P_1O$.
- Draw other segments P_iO and extend to points Q_i so that $Q_iO = P_iO$.
- Draw $\overline{Q_1A}$, $\overline{Q_1C}$, $\overline{Q_2A}$, $\overline{Q_2C}$, and so on.

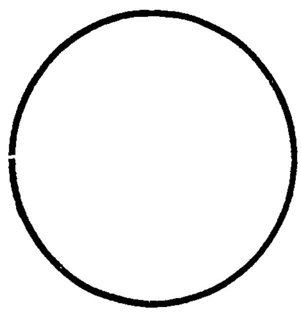
14. What kind of polygon is Q_1AP_1C, Q_2AP_2C, and so on? ______
    Why? ______
15. How does AC compare to Q_1P_1, Q_2P_2, and so on?
    ______
16. What part of the rectangle is each of the segments in question 15?
    ______
17. What figure starts to appear if you look at A, C, several points P_i, and several points Q_i? ______
18. The diagonals of the rectangles (for example, Q_1P_1 or AC) are what part of the circle? ______

Did you know that . . .

- any rhombus inscribed in a semicircle is a square?
- the measure of an inscribed angle is one-half the measure of the central angle that intercepts the same arc?
- inscribed angles that intercept the same or congruent arcs are equal in measure?
- a circle passes through the midpoints of the sides of a triangle, the feet of the perpendiculars from the vertices on the sides and the midpoints of the line segments between the vertices and the points of intersection of the altitudes? This circle is known as the nine-point circle.
- travel from a point A to a point B on the earth by plane follows what is known as the great circle route to minimize both fuel and time?

Can you . . .

- inscribe a square in a circle?
- prove that the opposite angles of a cyclic quadrilateral (a quadrilateral inscribed in a circle) are supplementary?
- find $m\angle AD_1B$ in figure 2 if $m\angle AOB = 88$?
- find a strategy to win a contest in which the goal is to construct as many pairs of perpendicular lines as you can in one minute using only a compass and straightedge?
- find the center of this circle using only the corner of a sheet of paper?



Solutions: (Angle measures are only approximations.)

1. The measure of each angle is 52°.
2. All angles have the same measure.
3. The measure of each angle is 128°.
4. All angles have the same measure.
5. The sum of the measures of the angles is 180; they are supplementary.
6. $m\angle AOB = 104°$.
7. The $m\angle AC_1B$, $m\angle AC_2B$, $m\angle AC_3B$, and $m\angle AC_4B$ are half of $m\angle AOB$.
8. diameter
9. semicircle
10. 180
11. The measure of each angle is 90°.
12. right angle
13. semicircle
14. rectangles. Answers will vary.
15. equal
16. diagonal
17. circle
18. diameter

Can you? $m\angle AD_1B = 136$

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Paper-punching Patterns

Follow the directions and discover some interesting numerical patterns.

- Punch a hole in a sheet of paper.
- Fold the paper in half and punch through the doubled sheet. (Make sure the second punch is different from the first.)
- Guess the number of holes in the paper if it is unfolded.
- Unfold the paper, check your guess, and record your results in table 1.
- Refold the paper as it was; fold it in half once more and punch through all the layers.
- Guess the total number of holes, check your guess, and record the results.
- Record the number of sections formed.
- Use the pattern you observe to extend the pattern beyond the values that you can actually punch.

Do your results look like these?

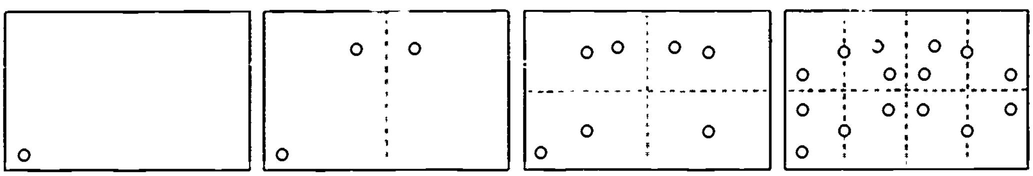


Table 1

Number of times folded in half	Number of sections	Number of holes
0	___	___
1	___	___
2	___	___
3	___	___
4	___	___
5	___	___
6	___	___
7	___	___
8	___	___

What do you think will happen if you vary the number of sections or punches on each turn? You'll have a chance to explore these questions in the following examples.

First, let's try changing the way we fold the paper:

- Punch a hole in a new sheet of paper.
- Fold the paper in thirds and punch all layers as before.
- Fold the paper in thirds a second time and punch through all the layers.
- Guess the total number of sections formed in the paper after each folding and guess the number of holes.
- Unfold the paper and record your results in table 2.
- Extend the pattern in table 2 to find values you can no longer fold or punch.

Do your results look like these?

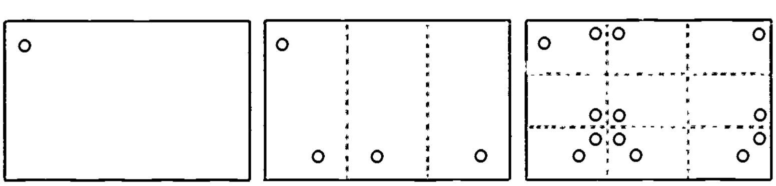


Table 2

Number of times folded in thirds	Number of sections	Number of holes
0	___	___
1	___	___
2	___	___
3	___	___
4	___	___
5	___	___
6	___	___
7	___	___
8	___	___

Do you see a relationship between the first and second columns of table 1 and table 2? Do you see a relationship between the second and third columns of table 1 and table 2? Are the patterns the same?

The editors wish to thank John Firkins, Gonzaga University, Spokane, WA 99258, for writing this issue of *NCTM Student Math Notes*.

Extending the Patterns

Data from tables 1 and 2 are recorded in a different way below to make one of the patterns clearer. Can you complete tables 3 and 4 for any whole number n?

How are the nth rows of tables 3 and 4 similar to each other? ______________________________

____

How are they different? ______________________________

____

Data from table 3 have been recorded in another way in table 5 to make a different pattern clearer. Can you complete this table for any n?

Table 3

Number of times folded in half	Number of sections	Number of punches	Number of holes
0	$\mathbf{1}=2^0$	1	$1=\mathbf{1}\cdot 1$
1	$\mathbf{2}=2^1$	1	$3=\mathbf{1}\cdot 1+\mathbf{2}\cdot 1$
2	$\mathbf{4}=2^2$	1	$7=\mathbf{1}\cdot 1+\mathbf{2}\cdot 1+\mathbf{4}\cdot 1$
3		1	
4		1	
.		.	
.		.	
.		.	
n		1	

Table 4

Number of times folded in thirds	Number of sections	Number of punches	Number of holes
0	$\mathbf{1}=3^0$	1	$1=\mathbf{1}\cdot 1$
1	$\mathbf{3}=3^1$	1	$4=\mathbf{1}\cdot 1+\mathbf{3}\cdot 1$
2	$\mathbf{9}=3^2$	1	$13=\mathbf{1}\cdot 1+\mathbf{3}\cdot 1+\mathbf{9}\cdot 1$
3		1	
4		1	
.		.	
.		.	
.		.	
n		1	

Table 5

Number of times folded in half	Number of sections	Number of punches	Number of holes
0	$1=2^0$	1	$1=2^1-1$
1	$2=2^1$	1	$3=2^2-1$
2	$4=2^2$	1	$7=2^3-1$
3	$8=2^3$	1	$15=2^4-1$
4			
5			
.			
.			
n			

Using powers of 3, see if you can develop a similar rule for table 4.

Suppose you fold a piece of paper into fourths or fifths and punch a hole each time you complete the folding. What would the results be? Complete tables 6 and 7.

Table 6

Number of times folded in fourths	Number of sections	Number of punches	Number of holes
0		1	1
1		1	
2		1	
3		1	
4		1	
.		.	
.		.	
.		.	
n		1	

Table 7

Number of times folded in fifths	Number of sections	Number of punches	Number of holes
0		1	1
1		1	
2		1	
3		1	
4		1	
.		.	
.		.	
.		.	
n		1	

More Patterns

Now we're going to change the number of punches:

- Punch a hole in a new sheet of paper.
- Fold it in half and punch it two more times.
- Guess how many holes you will see when you unfold the paper.
- Unfold the paper and record your findings in table 8.
- Refold the paper, fold it in half again, and punch three times through the paper; record the results.
- Extend the pattern.

Make up your own rule for folding and punching: ______________________________

____

Using your rule, punch holes in a new sheet of paper and record your results in table 9.

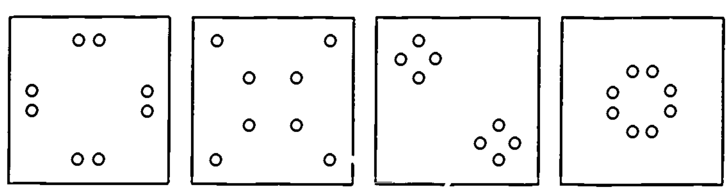
Table 8

Number of times folded in half	Number of sections	Number of punches	Number of holes
0	**1**	1	$1 = \mathbf{1} \cdot 1$
1	**2**	2	$5 = \mathbf{1} \cdot 1 + \mathbf{2} \cdot 2$
2	**4**	3	$17 = \mathbf{1} \cdot 1 + \mathbf{2} \cdot 2 + \mathbf{4} \cdot 3$
3			
4			
5			
.			
.			
.			
n			

Table 9

Number of times folded	Number of sections	Number of punches	Number of holes
0			
1			
2			
3			
4			
5			
.			
.			
.			
n			

Can you . . .



- fold a square sheet of paper and make one punch so that you will have each of the patterns at the right when you open it? (Use a separate piece of paper for each pattern.)
- recognize this pattern? Punch a hole in a new piece of paper. Fold it in half and punch through the same hole. Now fold it in thirds and again punch through the same hole. Continue to extend this pattern in your "mind's eye" by folding the paper in fourths, fifths, sixths, and so on, always punching through the same hole. What familiar number pattern is generated?
- describe a hole-punching activity to obtain the sequence 1, 5, 23, 119, . . . ?
- continue the pattern of punching and folding as seen in tables 5, 6, and 7 from fifths to sixths to sevenths and so on? What do you think will be the general formula for the number of holes after n foldings of the paper into $k$ths?

Did you know . . .

- that the sum of the divisors of some numbers can be determined by hole punching? The sum of the divisors of 8 is 15. The first hole-punching activity produced the same result. Write 8 as 2 to the third, the third power of the prime 2. The prime, 2, tells us to fold the paper in half at each step, and the power of the prime, 3, tells us to continue the hole-punching activity until we have folded the paper in half for the third time. Thus, if a number is a power of a prime, the foregoing activities calculate the sum of its divisors. The formula for computing the sum of the divisors of a power of a prime is given by

$$S = \frac{p^{k+1} - 1}{p - 1},$$

where p is the prime number and k is the power of the prime.

Solutions

Page 1

Table 1

times folded in half	0	1	2	3	4	5	6	7	8
sections	1	2	4	8	16	32	64	128	256
holes	1	3	7	15	31	63	127	255	511

Table 2

times folded in thirds	0	1	2	3	4	5	6	7	8
sections	1	3	9	27	81	243	729	2187	6561
holes	1	4	13	40	121	364	1093	3280	9841

Page 2

Rule for table 4:

sections	holes
3^n	$\frac{3^{n+1} - 1}{2}$

Table 5

sections	holes
2^n	$2^{n+1} - 1$

Table 6

folds	*sections*	*holes*
0	1	1
1	4	5
2	16	21
3	64	85
.	.	.
.	.	.
.	.	.
n	4^n	$\frac{4^{n+1} - 1}{3}$

Table 7

folds	*sections*	*holes*
0	1	1
1	5	6
2	25	31
3	125	156
.	.	.
.	.	.
.	.	.
n	5^n	$\frac{5^{n+1} - 1}{4}$

Page 3

Table 8

folds	*holes*
0	1
1	5
2	17
3	49
.	.
.	.
.	.
n	$n(2^{n+1}) + 1$

Can you . . .

Pattern generated—the factorials

Hole-punching activity A possible solution is to punch a hole in a new sheet of paper Fold it in half and punch it twice, fold it in thirds and punch it three times, and so on.

$$\frac{K^{n+1} - 1}{K - 1}$$

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